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Singularly Perturbed Boundary Value Problems for Systems of Tichonov's Type in Case of Exchange of Stabilities[1]

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We consider a system of ordinary differential equations consisting of a singularly perturbed scalar differential equation of second order and a scalar differential equation of first or second order and study a Neumann–Cauchy or a Neumann–Dirichlet problem. We assume that the degenerate equation has two intersecting solutions such that the standard theory for systems of Tichonov's type cannot be applied. We introduce the notation of a degenerate stable solution. By means of the technique of ordered lower and upper solutions we prove the existence of a solution of our problems near the degenerate stable solution for sufficiently small ε and determine its asymptotic behavior in ε. © 1999 Academic Press

Key Words: exchange of stabilities; singularly perturbed boundary value problems.

1. INTRODUCTION

The dynamics of fast bimolecular reactions can be modelled by means of singular singularly perturbed differential equations [8, 10]. If we try to reduce the order of this system we obtain a singularly perturbed differential system which can exhibit the property of exchange of stabilities. This phenomenon is characterized by the existence of intersecting solutions of the corresponding degenerate system which imply an exchange of stabilities of the families of equilibria of the associated system at these intersection

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points. Initial value problems for singularly perturbed systems in case of exchange of stabilities have been investigated by Lebovitz and Schaar [6] and others. Recently, Nefedov and Schneider [9] studied this problem by applying the asymptotic method of differential inequalities. This approach is based on well-known results of Chaplygin on differential inequalities [4], additionally the upper and lower solutions depend on the perturbation parameter which is assumed to tend to zero.

Singularly perturbed boundary value problems in case of exchange of stabilities have been considered only in special situations. In [1] a boundary value problem of this type has been studied for a scalar ordinary differential equation of second order. The essential result in that paper is the existence of a non-smooth limit solution; additionally, error estimates have been derived for the constructed asymptotic representation of the solution. The proofs are based on the application of the asymptotic method of differential inequalities justified by the results of Nagumo [7]. To get lower and upper solutions, the smoothing-procedure for non-smooth terms in the asymptotic expansion has effectively used (see also [2]).

In this paper, we apply a method developed in [1, 9] to a larger class of boundary value problems for singularly perturbed systems of Tichonov's type with fast and slow variables. Systems of such type play an important role in modelling processes with different time scales, especially they can be used to describe fast bimolecular reactions [10]. Thus, the results obtained in this paper can be used to investigate the behavior of reaction rates.

2. FORMULATION OF THE PROBLEM

Systems of differential equations containing "fast" and "slow" equations are called systems of Tichonov's type [12]. In what follows we consider such systems consisting of a "fast" differential equation of second order

$$\varepsilon^2 u'' = g(u, v, x, \varepsilon), \tag{2.1}$$

here and in the sequel we denote by "$'$" the differentiation with respect to x, and of a "slow" differential equation either of first order

$$v' = f(u, v, x, \varepsilon) \tag{2.2}$$

or of second order

$$v'' = f(u, v, x, \varepsilon). \tag{2.3}$$

Let I_{ε_0} be the interval $I_{\varepsilon_0} := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_0\}$ where $0 < \varepsilon_0 \ll 1$. Let $D := R \times R \times (0, 1) \times I_{\varepsilon_0}$. Concerning the functions f and g we suppose

(C_0) $f, g \in C^2_{\{uvx\varepsilon\}}(D, R)$ where all derivatives are continuous in the closure of D.

We look for a solution (u, v) of Eqs. (2.1)–(2.3) whose u-component satisfies the no-flux condition

$$u'(0) = u'(1) = 0, \tag{2.4}$$

whereas the v-component is assumed to obey either the initial condition

$$v(0) = v^0 \tag{2.5}$$

in case of Eq. (2.2) or the boundary conditions

$$v(0) = v^0, \qquad v(1) = v^1 \tag{2.6}$$

in case of Eq. (2.3).

The restriction on this type of boundary conditions is not essential. We denote the boundary-initial value problem (2.1), (2.2), (2.4), (2.5) as (BIVP), and the boundary value problem (2.1), (2.3), (2.4), (2.6) as (BVP). We consider these problems under the following assumptions:

(C_1) The degenerate equation

$$g(u, v, x, 0) = 0 \tag{2.7}$$

has two solutions $u = \varphi_1(v, x)$ and $u = \varphi_2(v, x)$ with the same smoothness as g. In the (v, x)-plane there is a continuous curve $v = v_0(x)$, $0 \leqslant x \leqslant 1$, such that for $0 \leqslant x \leqslant 1$

$$\begin{aligned} \varphi_1(v, x) &> \varphi_2(v, x) && \text{for} \quad v < v_0(x), \\ \varphi_1(v_0(x), x) &\equiv \varphi_2(v_0(x), x) && \text{for} \quad 0 \leqslant x \leqslant 1, \\ \varphi_1(v, x) &< \varphi_2(v, x) && \text{for} \quad v > v_0(x). \end{aligned} \tag{2.8}$$

From assumption (C_1) it follows that the solutions $u = \varphi_1(v, x)$ and $u = \varphi_2(v, x)$ intersect at a curve whose projection into the (v, x)-plane is the curve $v = v_0(x)$. This property distinguishes the problem under consideration from the standard case treated in Tichonov's theorem (see [12]) and its analogs where only isolated solutions of the degenerate equation (2.7) are considered.

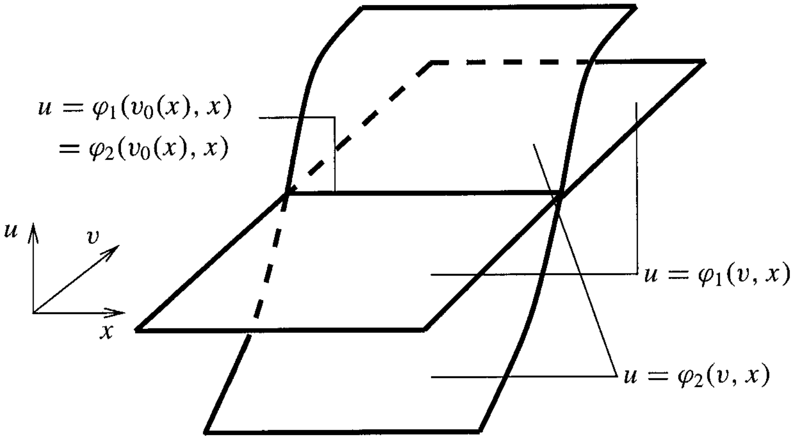

FIG. 1. Intersection of the roots of $g = 0$.

(C_2) For $0 \leqslant x \leqslant 1$ we suppose

$$
\begin{aligned}
&g_u(\varphi_1(v, x), v, x, 0) > 0 \qquad \text{for} \quad v < v_0(x),\\
&g_u(\varphi_1(v, x), v, x, 0) < 0 \qquad \text{for} \quad v > v_0(x),\\
&g_u(\varphi_2(v, x), v, x, 0) < 0 \qquad \text{for} \quad v < v_0(x),\\
&g_u(\varphi_2(v, x), v, x, 0) > 0 \qquad \text{for} \quad v > v_0(x).
\end{aligned}
$$

From assumption (C_2) we obtain that $g_u(u, v, x, 0)$ changes its sign on each solution $u = \varphi_1(v, x)$ and $u = \varphi_2(v, x)$ when v passes the point $v_0(x)$ on the curve $v = v_0(x)$ for $0 \leqslant x \leqslant 1$. Hence, we have

$$
\begin{aligned}
g_u(\varphi_1(v_0(x), x), v_0(x), x) &\equiv g_u(\varphi_2(v_0(x), x), v_0(x), x)\\
&\equiv 0 \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \qquad (2.9)
\end{aligned}
$$

This property implies a change of the qualitative behavior of the equilibria $u = \varphi_1(v, x)$ and $u = \varphi_2(v, x)$ of the corresponding associated equation

$$\frac{d^2u}{d\xi^2} = g(u, v, x, 0)$$

where v and x are considered as parameters.

The simplest example of a function g satisfying the hypotheses (C_1) and (C_2) is a function quadratic in u, $g(u, v, x, 0) \equiv [u - \varphi_1(v, x)][u - \varphi_2(v, x)]$, provided φ_1 and φ_2 fulfil the relation (2.8).

Further hypotheses are introduced in studying each of the problems characterized above.

3. THE DEGENERATE STABLE SOLUTION FOR THE BOUNDARY-INITIAL VALUE PROBLEM

Consider the boundary-initial value problem

$$\begin{aligned} \varepsilon^2 u'' &= g(u, v, x, \varepsilon), \\ v' &= f(u, v, x, \varepsilon), \\ u'(0) &= u'(1) = 0, \\ v(0) &= v^0. \end{aligned} \tag{BIVP}$$

First we study the case that the initial value v^0 in (2.5) for the differential equation (2.2) satisfies $v^0 < v_0(0)$. In that case, we replace u in the right hand side of (2.2) by the function $\varphi_1(v, x)$ and consider the initial value problem

$$v' = f(\varphi_1(v, x), v, x, 0), \quad v(0) = v^0, \qquad 0 \leqslant x < 1. \tag{3.1}$$

We assume (3.1) has a solution $v = v_1(x)$ intersecting the curve $v = v_0(x)$ for $x = x_0 < 1$; that is v_1 satisfies

$$v_1(x) < v_0(x) \qquad \text{for} \quad 0 \leqslant x < x_0, \quad v_1(x_0) = v_0(x_0). \tag{3.2}$$

Furthermore, for $x \geqslant x_0$ we study the initial value problem

$$v' = f(\varphi_2(v, x), v, x, 0), \quad v(x_0) = v_0(x_0), \qquad x_0 \leqslant x \leqslant 1. \tag{3.3}$$

We assume it has a solution $v = v_2(x)$ satisfying

$$v_2(x) > v_0(x) \qquad \text{for} \quad x_0 < x \leqslant 1. \tag{3.4}$$

Now, we introduce the functions $\hat{v}(x)$ and $\hat{u}(x)$ by

$$\begin{aligned} \hat{v}(x) &= \begin{cases} v_1(x) & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ v_2(x) & \text{for} \quad x_0 \leqslant x \leqslant 1, \end{cases} \\ \hat{u}(x) &= \begin{cases} \varphi_1(v_1(x), x) \equiv u_1(x) & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ \varphi_2(v_2(x), x) \equiv u_2(x) & \text{for} \quad x_0 \leqslant x \leqslant 1. \end{cases} \end{aligned} \tag{3.5}$$

The function $\hat{v}(x)$ is continuously differentiable in $[0, 1]$, i.e., $\hat{v}(x)$ is a classical solution of the initial value problem

$$v' = f(\varphi(v, x), v, x, 0), \quad v(0) = v^0, \qquad 0 \leqslant x \leqslant 1 \tag{3.6}$$

where

$$\varphi(v, x) = \begin{cases} \varphi_1(v, x) & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ \varphi_2(v, x) & \text{for} \quad x_0 \leqslant x \leqslant 1. \end{cases} \tag{3.7}$$

In contrast to $\hat{v}(x)$, the function $\hat{u}(x)$ is less smooth. It is continuous in $[0, 1]$ and has continuous first and second derivatives except at $x = x_0$ where both derivatives can have a discontinuity. From (2.8) we get

$$\hat{u}'(x_0 - 0) = u_1'(x_0) \leqslant u_2'(x_0) = \hat{u}'(x_0 + 0). \tag{3.8}$$

The pair of functions $(\hat{u}(x), \hat{v}(x))$ is referred to as the degenerate stable solution. It is constructed by means of the solutions $u = \varphi_1(v, x)$ and $u = \varphi_2(v, x)$ of the degenerate equation (2.7).

Summarizing the considerations above we introduce the following hypothesis.

(I_1) The initial value problem (3.6) has a solution $\hat{v}(x)$ satisfying (3.2) and (3.4).

Remark 1. In order to construct $\hat{v}(x)$ in case $v^0 > v_0(0)$, we have to use the function $\varphi_2(v, x)$ on the interval $[0, x_0]$ and the function $\varphi_1(v, x)$ on the interval $[x_0, 1]$. In case $v^0 = v_0(0)$ we have to compare $v_1'(0) = f(\varphi_i(v^0, 0), v^0, 0, 0)$ and $v_0'(0)$. If $v_1'(0) < v_0'(0)$ $(v_1'(0) > v_0'(0))$ we use the function $\varphi_1(v, x)$ $(\varphi_2(v, x))$ on the interval $[0, x_0]$ and the function $\varphi_2(v, x)$ $(\varphi_1(v, x))$ on the interval $[x_0, 1]$. If additionally $v_1'(0) = v_0'(0)$ holds we have to compare $v_1''(0)$ and $v_0''(0)$, and so on.

Remark 2. The curve $v = \hat{v}(x)$ intersects the curve $v = v_0(x)$ at a unique point, namely at $(x_0, v_0(x_0))$. The more general case of several intersection points can be also treated. In crossing each intersection point we have to

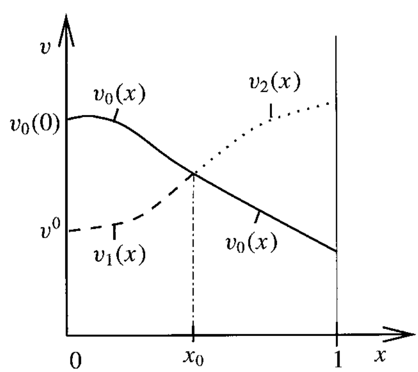

FIG. 2. Location of the solutions $v_1(x)$ and $v_2(x)$ with respect to $v_0(x)$.

replace one of the functions φ_1 and φ_2 in the right hand side of (3.6) by the other one.

In the next section we shall prove that under some additional assumptions the degenerate stable solution is the limit of the solution $(u(x, \varepsilon), v(x, \varepsilon))$ of (BIVP) as ε tends to zero.

4. THE ASYMPTOTIC LIMIT IN THE BOUNDARY-INITIAL VALUE PROBLEM

For the sequel it is convenient to introduce the notation $\hat{g}_u(x) := g_u(\hat{u}(x), \hat{v}(x), x, 0)$. Analogously we use a similar notation for other derivatives of the functions g and f.

Note that we get from assumption (C_2)

$$\hat{g}_u(x) > 0 \qquad \text{for} \quad x \neq x_0, \quad \text{and} \quad \hat{g}_u(x_0) = 0. \tag{4.1}$$

Concerning the second derivative we assume

(I_2) $\hat{g}_{uu}(x_0) > 0$.

In case that $g(u, v, x, 0)$ has the form $g(u, v, x, 0) = (u - \varphi_1(v, x))(u - \varphi_2(v, x))$ this condition is fulfilled.

The following assumption concerns the dependence of the function g on the parameter ε. The cases that g is independent of ε and that g depends on ε require a separate treatment (see [1]). Here, we consider the case that g depends on ε. In that case the sign of the derivative $\hat{g}_\varepsilon(x)$ at $x = x_0$ plays a crucial role (see also [1]).

(I_3) $\hat{g}_\varepsilon(x_0) < 0$.

THEOREM 1. *Assume the hypotheses* (C_0)–(C_2) and (I_1)–(I_3) *are valid. Then, for sufficiently small* ε, *the boundary initial value problem* (BIVP) *has a solution* $(u(x, \varepsilon), v(x, \varepsilon))$ *satisfying for* $x \in [0, 1]$

$$u(x, \varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}), \qquad v(x, \varepsilon) = \hat{v}(x) + O(\sqrt{\varepsilon}). \tag{4.2}$$

Proof. The proof is based on the technique of differential inequalities. For convenience we recall the notion of ordered upper and lower solutions.

Two pairs of functions $(\underline{U}(x, \varepsilon), \underline{V}(x, \varepsilon))$ and $(\overline{U}(x, \varepsilon), \overline{V}(x, \varepsilon))$ are called ordered lower and upper solutions of (BIVP) respectively iff they satisfy the following conditions:

1^0. $\underline{U}(x, \varepsilon) \leqslant \bar{U}(x, \varepsilon)$, $\underline{V}(x, \varepsilon) \leqslant \bar{V}(x, \varepsilon)$ for $0 \leqslant x \leqslant 1$;

2^0. $L_\varepsilon(\underline{U}, v) \equiv \varepsilon^2 \underline{U}'' - g(\underline{U}, v, x, \varepsilon) \geqslant 0$, and $L_\varepsilon(\bar{U}, v) \leqslant 0$ for $\{0 < x < 1, \underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)\}$;
$M_\varepsilon(u, \underline{V}) \equiv \underline{V}' - f(u, \underline{V}, x, \varepsilon) \leqslant 0$ and $M_\varepsilon(u, \bar{V}) \geqslant 0$ for $\{0 < x < 1, \underline{U}(x, \varepsilon) \leqslant u \leqslant \bar{U}(x, \varepsilon)\}$;

3^0. $\underline{U}'(0, \varepsilon) \geqslant 0 \geqslant \bar{U}'(0, \varepsilon)$, $\underline{U}'(1, \varepsilon) \leqslant 0 \leqslant \bar{U}'(1, \varepsilon)$, $\underline{V}(0, \varepsilon) \leqslant v^0 \leqslant \bar{V}(0, \varepsilon)$.

It is known [11] that the existence of ordered lower and upper solutions implies the existence of a solution $(u(x, \varepsilon), v(x, \varepsilon))$ of (BIVP) obeying

$$\underline{U}(x, \varepsilon) \leqslant u(x, \varepsilon) \leqslant \bar{U}(x, \varepsilon),\ \underline{V}(x, \varepsilon) \leqslant v(x, \varepsilon) \leqslant \bar{V}(x, \varepsilon) \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{4.3}$$

To construct ordered lower and upper solutions we use the degenerate stable solution $(\hat{u}(x), \hat{v}(x))$. By (3.8) the derivative of $\hat{u}(x)$ has a non-negative jump at $x = x_0$: $\hat{u}'(x_0 + 0) - \hat{u}'(x_0 - 0) \geqslant 0$. We note that the first derivative of the lower solution $\underline{U}(x, \varepsilon)$ can have a positive jump at $x = x_0$, but not the first derivative of the the upper solution $\bar{U}(x, \varepsilon)$ (see [3, 5]). Hence, to be able to construct $\bar{U}(x, \varepsilon)$ by using $\hat{u}(x)$ we introduce a smoothing procedure developed in [2] for problems with non-smooth terms in the asymptotic expansions.

Let

$$\xi = \frac{x - x_0}{\varepsilon^\alpha}$$

where α is any number of the interval (0.5, 1), let

$$\omega(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\xi} e^{-s^2}\, ds.$$

Obviously, we have $\omega(-\infty) = 0$, $\omega(+\infty) = 1$.

We extend the function $v_1(x)$ for $x > x_0$ as solution of (3.1) with the initial condition $v_1(x_0) = v_0(x_0)$ (see (3.2)), and the function $v_2(x)$ for $x < x_0$ as solution of (3.3). This permits to extend smoothly the function $u_1(x) \equiv \varphi_1(v_1(x), x)$ for $x > x_0$, and the function $u_2(x) \equiv \varphi_2(v_2(x), x)$ for $x < x_0$. We put for $0 \leqslant x \leqslant 1$

$$\tilde{u}(x) = u_1(x)\, \omega(-\xi) + u_2(x)\, \omega(\xi). \tag{4.4}$$

The function $\tilde{u}(x)$ is smooth in [0, 1], and it holds

$$\tilde{u}(x) = \hat{u}(x) + 0(\varepsilon^\alpha) \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{4.5}$$

Now we construct lower and upper solutions for (BIVP) in the form

$$\underline{U}(x,\varepsilon)\equiv \hat{u}(x)-\sqrt{\varepsilon}\,\sigma\,e^{\lambda(x-x_0)}-\varepsilon^{\alpha}z(x,\varepsilon),\quad \underline{V}(x,\varepsilon)=\hat{v}(x)-\sqrt{\varepsilon}\,\sigma^2\,e^{\lambda(x-x_0)},$$
$$\bar{U}(x,\varepsilon)\equiv \tilde{u}(x)+\sqrt{\varepsilon}\,\eta\,e^{\lambda(x-x_0)}+\varepsilon^{\alpha}z(x,\varepsilon),\quad \bar{V}(x,\varepsilon)=\hat{v}(x)+\sqrt{\varepsilon}\,\sigma^2\,e^{\lambda(x-x_0)},\tag{4.6}$$

where $z(x,\varepsilon)\equiv e^{-\kappa x/\varepsilon^{\alpha}}+e^{-\kappa(1-x)/\varepsilon^{\alpha}}$, and $\sigma,\lambda,\kappa,\eta$ are positive numbers. We shall determine these numbers such that the functions $\underline{U}$, $\underline{V}$, $\bar{U}$, $\bar{V}$ satisfy the conditions 1^0–3^0 for sufficiently small ε that is, they are ordered lower and upper solutions.

Condition 1^0 is obviously fulfilled for any positive $\sigma,\lambda,\kappa,\eta$ and sufficiently small ε. The last (double) inequality in 3^0 is also fulfilled for any positive σ and λ since we have $\hat{v}(0)=v_1(0)=v^0$. The remaining inequalities in 3^0 are satisfied if κ is sufficiently large. For example, from $\underline{U}'(0,\varepsilon)=\hat{u}'(0)+\kappa+O(\sqrt{\varepsilon})$ it follows $\underline{U}'(0,\varepsilon)\geqslant 0$ for κ sufficiently large.

Now we consider the inequalities in 2^0. Firstly we are concerned with $L_\varepsilon(\bar{U},v)$. By expanding $g(\bar{U}(x,\varepsilon),v,x,\varepsilon)$ into a Taylor series at $(\hat{u}(x),\hat{v}(x),x,0)$ and taking into account the relations (4.5) and $\hat{g}(x)\equiv 0$ we get

$$\begin{aligned}L_\varepsilon(\bar{U},v)=\Big\{&\varepsilon^2[u_1''(x)\,\omega(-\xi)+u_2''(x)\,\omega(\xi)]+\varepsilon^{2-\alpha}\frac{2}{\sqrt{\pi}}[u_2'(x)-u_1'(x)]\,e^{-\xi^2}\\&+\varepsilon^{2-2\alpha}\frac{2}{\sqrt{\pi}}[u_1(x)-u_2(x)]\,\xi e^{-\xi^2}\\&+\varepsilon^{5/2}\eta\lambda^2e^{\lambda(x-x_0)}+\varepsilon^{2-\alpha}\kappa^2z(x,\varepsilon)\Big\}\\&-\Big[\hat{g}_u(x)(\sqrt{\varepsilon}\,\eta e^{\lambda(x-x_0)}+O(\varepsilon^{\alpha}))+\hat{g}_v(x)(v-\hat{v}(x))+\hat{g}_\varepsilon(x)\,\varepsilon\\&+\frac{1}{2}\{\hat{g}_{uu}(x)(\sqrt{\varepsilon}\,\eta e^{\lambda(x-x_0)}+O(\varepsilon^{\alpha}))^2+2\hat{g}_{uv}(x)(\sqrt{\varepsilon}\,\eta e^{\lambda(x-x_0)}\\&+O(\varepsilon^{\alpha}))(v-\hat{v}(x))+\hat{g}_{vv}(x)(v-\hat{v}(x))^2\}+o(\varepsilon)\Big].\end{aligned}$$

Since $[u_1(x)-u_2(x)]\,\xi e^{-\xi^2}=O(\varepsilon^{\alpha})$ the expression in the first braces is of order $\varepsilon^{2-\alpha}$, and satisfies also $o(\varepsilon)$ because of $\frac{1}{2}<\alpha<1$. Let us rewrite the expression in the square brackets. We set

$$v-\hat{v}(x)=\sqrt{\varepsilon}\,\sigma^2e^{\lambda(x-x_0)}w.$$

Then we have $-1 \leqslant w \leqslant 1$ for $\underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)$. We will also exploit the relationship

$$\hat{g}_v(x) = -\hat{g}_u(x)\, \varphi_v(\hat{v}(x), x)$$

where the function $\varphi(v, x)$ is defined in (3.7). We get this equality by differentiating $g(\varphi(v, x), v, x, 0) \equiv 0$ with respect to v. Consequently, we obtain

$$L_\varepsilon(\bar{U}, v) = -\sqrt{\varepsilon}\, \hat{g}_u(x)\, e^{\lambda(x-x_0)}(\eta - \hat{\varphi}_v(x)\, \sigma^2 w + O(\varepsilon^{\alpha-1/2})) - \varepsilon \hat{g}_\varepsilon(x)$$
$$- \tfrac{1}{2} \varepsilon e^{2\lambda(x-x_0)}(\hat{g}_{uu}(x)\, \eta^2 + 2\hat{g}_{uv}(x)\, \eta\sigma^2 w + \hat{g}_{vv}(x)\, \sigma^4 w^2) + o(\varepsilon). \tag{4.7}$$

For sufficiently small σ and ε we have

$$\eta - \hat{\varphi}_v(x)\, \sigma^2 w + O(\varepsilon^{\alpha-1/2}) \geqslant \frac{\eta}{2}.$$

From (4.1) we can conclude that the first term on the right hand side of (4.7) is not greater than zero for $x \in [0, 1]$. The second term $-\varepsilon \hat{g}_\varepsilon(x)$ is positive at x_0 according to hypothesis (I_3). The third term is also of order ε as the second one. For sufficiently small σ it is negative at $x = x_0$ by hypothesis (I_2). Moreover, for sufficiently large η the sum of the second and third terms is negative at $x = x_0$, and also in some small ν-neighborhood of x_0. Thus, in this neighborhood the inequality $L_\varepsilon(\bar{U}, v) \leqslant 0$ is valid for $\underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)$ by hypothesis (I_2). Outside this ν-neighborhood of x_0 we have $\hat{g}_u(x) \geqslant c > 0$. Hence, the dominant term on the right hand side in (4.7) is the first one possessing the order $\sqrt{\varepsilon}$. For sufficiently small ε, this term guarantees the validity of the inequality $L_\varepsilon(\bar{U}, v) \leqslant 0$ for $|x - x_0| \geqslant \nu$, $\underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)$. Consequently, the inequality $L_\varepsilon(\bar{U}, v) \leqslant 0$ in condition 1^0 is valid under our assumptions.

Analogously to the derivation of (4.7), we get for $L_\varepsilon(\underline{U}, v)$

$$L_\varepsilon(\underline{U}, v) = \sqrt{\varepsilon}\, \hat{g}_u(x)\, \sigma e^{\lambda(x-x_0)}(1 - \hat{\varphi}_v(x)\, \sigma w + O(\varepsilon^{\alpha-1/2}))$$
$$- \varepsilon \hat{g}_\varepsilon(x) - \frac{\varepsilon}{2} \sigma^2 e^{2\lambda(x-x_0)}(\hat{g}_{uu}(x) + 2\hat{g}_{uv}(x)\, \sigma w + \hat{g}_{vv}(x)\, \sigma^2 w^2) + o(\varepsilon). \tag{4.8}$$

For sufficiently small σ and ε we have

$$1 - \sigma \hat{\varphi}_v(x)\, w + O(\varepsilon^{\alpha-1/2}) \geqslant \tfrac{1}{2}.$$

Therefore, the first term on the right hand side in (4.8) is not less than zero. The second term $-\varepsilon \hat{g}_\varepsilon(x)$ is positive at $x = x_0$ due to assumption (I_3). For sufficiently small σ, the third term is negative at $x = x_0$ by hypothesis (I_2). Obviously, for sufficiently small σ, the sum of the second and of the third term is positive at $x = x_0$, and also in some ν-neighborhood of x_0. Thus, for sufficiently small σ and ε, the inequality $L_\varepsilon(\underline{U}, v) \geqslant 0$ holds for $\underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)$ and $|x - x_0| \leqslant \nu$.

Outside this neighborhood the first term is of order $\sqrt{\varepsilon}$ and dominates the other ones, and the inequality $L_\varepsilon(\underline{U}, v) \geqslant 0$ is fulfilled for sufficiently small ε. Hence, the inequality $L_\varepsilon(\underline{U}, v) \geqslant 0$ holds for $\underline{V}(x, \varepsilon) \leqslant v \leqslant \bar{V}(x, \varepsilon)$ and $0 \leqslant x \leqslant 1$.

For $M_\varepsilon(u, \underline{V})$ we get

$$M_\varepsilon(u, \underline{V}) = \{\hat{v}'(x) - \sqrt{\varepsilon}\,\sigma^2 \lambda e^{\lambda(x - x_0)}\}$$
$$- [\hat{f}(x) + \hat{f}_u(x)(u - \hat{u}(x)) - \hat{f}_v(x)\sqrt{\varepsilon}\,\sigma^2 e^{\lambda(x - x_0)} + O(\varepsilon)].$$

If we set $u - \hat{u}(x) = \sqrt{\varepsilon}\, e^{\lambda(x - x_0)} w$ then we have $-\sigma + O(\varepsilon^{\alpha - 1/2}) \leqslant w \leqslant \eta + O(\varepsilon^{\alpha - 1/2})$ for $\underline{U}(x, \varepsilon) \leqslant u \leqslant \bar{U}(x, \varepsilon)$. Taking into account $\hat{v}'(x) = \hat{f}(x)$ we obtain

$$M_\varepsilon(u, \underline{V}) = -\sqrt{\varepsilon}\, e^{\lambda(x - x_0)}[\sigma^2 \lambda + \hat{f}_u(x)\, w - \hat{f}_v(x)\, \sigma^2] + O(\varepsilon).$$

For sufficiently large λ, the expression in the square brackets is positive. Thus, for sufficiently small ε we have $M_\varepsilon(u, \underline{V}) \leqslant 0$ for $0 \leqslant x \leqslant 1$, $\underline{U}(x, \varepsilon) \leqslant u \leqslant \bar{U}(x, \varepsilon)$.

Analogously, the inequality

$$M_\varepsilon(u, \bar{V}) \geqslant 0 \qquad \text{for} \quad 0 \leqslant x \leqslant 1, \qquad \underline{U}(x, \varepsilon) \leqslant u \leqslant \bar{U}(x, \varepsilon)$$

can be established. Consequently, we have proved that the functions $(\underline{U}(x, \varepsilon), \underline{V}(x, \varepsilon))$ and $(\bar{U}(x, \varepsilon), \bar{V}(x, \varepsilon))$ defined in (4.6) satisfy the conditions 1^0–3^0 for appropriately chosen constants that is, they are ordered lower and upper solutions for (BIVP). Hence, there exists a solution $(u(x, \varepsilon), v(x, \varepsilon))$ of (BIVP) satisfying the inequalities (4.3). By (4.6) we have

$$\underline{U}(x, \varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}), \qquad \bar{U}(x, \varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}),$$
$$\underline{V}(x, \varepsilon) = \hat{v}(x) + O(\sqrt{\varepsilon}), \qquad \bar{V}(x, \varepsilon) = \hat{v}(x) + O(\sqrt{\varepsilon}).$$

Hence, from (4.3) we get the inequalities (4.2). This completes the proof of Theorem 4.

Remark 3. From (4.2) we obtain

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x), \qquad \lim_{\varepsilon \to 0} v(x, \varepsilon) = \hat{v}(x)$$

that is, the degenerate stable solution is the limit of the solution of (BIVP).

Remark 4. Outside any small but fixed ν-neighborhood of x_0 we can derive a higher order asymptotic expansion of the solution $(u(x, \varepsilon), v(x, \varepsilon))$ of (BIVP); for example (see [1]) we have

$$u(x, \varepsilon) = \hat{u}(x) + O(\varepsilon), \qquad v(x, \varepsilon) = \hat{v}(x) + O(\varepsilon). \tag{4.9}$$

Remark 5. The solution $(u(x, \varepsilon), v(x, \varepsilon))$ is not necessarily unique in the region bounded by the ordered lower and upper solutions [11] but all solution are located in an $O(\sqrt{\varepsilon})$-neighborhood of the degenerate stable solution.

Remark 6. If instead of hypothesis (I_3) the inequality $\hat{g}_\varepsilon(x_0) > 0$ holds then there can arise the case that (BIVP) has no solution converging to the degenerate stable solution $(\hat{u}(x), \hat{v}(x))$ as $\varepsilon \to 0$ (see the example in [1]).

Remark 7. As we already mentioned above, the case when the functions g and f do not depend on ε requires a separate treatment. We shall consider this case in more details in a forthcoming paper.

5. THE DEGENERATE STABLE SOLUTION FOR THE BOUNDARY VALUE PROBLEM (BVP)

Consider the boundary value problem

$$\begin{aligned} \varepsilon^2 u'' &= g(u, v, x, \varepsilon), \\ v'' &= f(u, v, x, \varepsilon), \\ u'(0) &= u'(1) = 0, \\ v(0) &= v^0, \qquad v(1) = v^1. \end{aligned} \tag{BVP}$$

We preserve hypotheses (C_0)–(C_2) from Section 2 and study first the case $v^0 < v_0(0)$, $v^1 > v_0(1)$. To construct the corresponding degenerate stable solution we consider the boundary value problems

$$v'' = f(\varphi_1(v, x), v, x, 0), \quad 0 \leqslant x \leqslant x_0, \quad v(0) = v^0, \quad v(x_0) = v_0(x_0), \tag{5.1}$$

$$v'' = f(\varphi_2(v, x), v, x, 0), \quad x_0 \leqslant x \leqslant 1, \quad v(x_0) = v_0(x_0), \quad v(1) = v^1. \tag{5.2}$$

For the sequel we need the assumption

(B_1) There exists a $x_0 \in (0, 1)$ such that the boundary value problems (5.1) and (5.2) have solutions $v_1(x)$ and $v_2(x)$ respectively satisfying

$$v_1(x) < v_0(x) \qquad \text{for} \quad 0 \leqslant x < x_0,$$
$$v_2(x) > v_0(x) \qquad \text{for} \quad x_0 < x \leqslant 1,$$
$$v_1'(x_0) = v_2'(x_0).$$

We introduce the functions $\hat{v}(x)$ and $\hat{u}(x)$ by

$$\hat{v}(x) = \begin{cases} v_1(x) & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ v_2(x) & \text{for} \quad x_0 \leqslant x \leqslant 1, \end{cases}$$

$$\hat{u}(x) = \begin{cases} \varphi_1(v_1(x), x) \equiv u_1(x) & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ \varphi_2(v_2(x), x) \equiv u_2(x) & \text{for} \quad x_0 \leqslant x \leqslant 1. \end{cases}$$

The function $\hat{u}(x)$ has the same properties as the corresponding one in Section 3, in particular, its first derivate satisfies at $x = x_0$ inequality (3.8); according to hypothesis (B_1), the function $v(x)$ is a classic solution (twice continuously differentiable in (0, 1)) of the boundary value problem

$$v'' = f(\varphi(v, x), v, x, 0), \qquad v(0) = v^0, \quad v(1) = v^1,$$

where $\varphi(v, x)$ is defined by (3.7).

The pair of functions $(\hat{u}(x), \hat{v}(x))$ is referred to as the degenerate stable solution of (BVP).

6. THE ASYMPTOTIC LIMIT IN THE BOUNDARY VALUE PROBLEM

To derive the following results about existence and asymptotic behavior of a solution of (BVP) we introduce the assumptions

(B_2)

$$\hat{g}_\varepsilon(x_0) < 0.$$

(B_3) There are positive numbers β and μ obeying the inequalities

$$\hat{\varphi}_v(x) \equiv \varphi_v(\hat{v}(x), x) < \beta \qquad \text{for} \quad 0 \leqslant x \leqslant 1, \tag{6.1}$$

$$\hat{g}_{uu}(x_0)\, \beta^2 + 2\hat{g}_{uv}(x_0)\, \beta + \hat{g}_{vv}(x_0) > 0, \tag{6.2}$$

$$\hat{f}_u(x)\, \beta + \hat{f}_v(x) \geqslant -\pi^2 + \mu \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{6.3}$$

(B_4) The function $g(u, v, x, \varepsilon)$ is non-increasing in v for fixed u, x, ε, and the function $f(u, v, x, \varepsilon)$ is non-increasing in u for fixed v, x, ε in some neighborhood (which will be specified after introducing lower and upper solutions) of the degenerate stable solution for sufficiently small ε.

Assumption (B_4) says that the vector function (g, f) is quasi-monotone in some neighborhood of the degenerate stable solution.

THEOREM 2. *Assume the hypotheses* (C_0)–(C_2) *and* (B_1)–(B4) *are valid. Then, for sufficiently small* ε, *the boundary value problem* (BVP) *has a solution* $(u(x, \varepsilon), (x, \varepsilon))$ *satisfying for* $x \in [0, 1]$

$$u(x, \varepsilon) = \hat{u}(x) + O(\sqrt{\varepsilon}), \qquad v(x, \varepsilon) = \hat{v}(x) + O(\sqrt{\varepsilon}). \tag{6.4}$$

Proof. Again we use the method of differential inequalities. Since Eq. (2.3) is a second order equation, the conditions for ordered lower $(\underline{U}(x, \varepsilon), \underline{V}(x, \varepsilon))$ and upper $(\bar{U}(x, \varepsilon), \bar{V}(x, \varepsilon))$ solutions partly change. Condition 1^0 remains the same. In 2^0, the inequalities concerning Eq. (2.1) also are preserved, whereas the inequalities concerning Eq. (2.3) change as follows

$$\begin{aligned} &M_\varepsilon(u, \underline{V}) \equiv \underline{V}'' - f(u, \underline{V}(x, \varepsilon), x, \varepsilon) \geqslant 0, \\ &M_\varepsilon(u, \bar{V}) \leqslant 0 \qquad \text{for} \quad \{0 \leqslant x \leqslant 1,\ \underline{U}(x, \varepsilon) \leqslant u \leqslant \bar{U}(x, \varepsilon)\}. \end{aligned} \tag{6.5}$$

Finally, the following inequalities have to be added to the conditions in 3^0:

$$\underline{V}(1, \varepsilon) \leqslant v^1 \leqslant \bar{V}(1, \varepsilon).$$

The existence of ordered lower and upper solutions implies the existence of a solution $(u(x, \varepsilon), v(x, \varepsilon))$ of the boundary value problem obeying the inequalities (4.3).

As in the proof of Theorem 1 we use the stable degenerate solution $(\hat{u}(x, \varepsilon), \hat{v}(x, \varepsilon))$ to construct ordered lower and upper solutions. Since the derivative of $\hat{u}(x)$ at $x = x_0$ has a non-negative jump, we use the smoothed function $\tilde{u}(x)$ defined in (4.4) to construct $\bar{U}(x, \varepsilon)$, where we choose $\alpha = 1$. Hence, we have

$$\tilde{u}(x, \varepsilon) = \hat{u}(x) + O(\varepsilon). \tag{6.6}$$

(Note, that also in the proof of Theorem 1 we could set $\alpha = 1$ without great changes in the proof.)

We put

$$\begin{aligned} &\underline{U}(x, \varepsilon) = \hat{u}(x) - \varepsilon\beta\gamma h(x) - \varepsilon z(x, \varepsilon), && \underline{V}(x, \varepsilon) = \hat{v}(x) - \varepsilon\gamma h(x), \\ &\bar{U}(x, \varepsilon) = \tilde{u}(x) + \sqrt{\varepsilon}\,\beta\gamma h(x) + \varepsilon z(x, \varepsilon), && \bar{V}(x, \varepsilon) = \hat{v}(x) + \sqrt{\varepsilon}\,\gamma h(x) \end{aligned} \tag{6.7}$$

where β is the same constant as in hypothesis (B_3), z and h are defined as follows

$$z(x, \varepsilon) := e^{-\kappa x/\varepsilon} + e^{-\kappa[(1-x)/\varepsilon]}, \qquad h(x) := \sin \frac{\pi(x+\delta)}{1+2\delta}, \tag{6.8}$$

κ, δ and γ are positive numbers to be chosen later appropriately such that the conditions 1^0–3^0 are fulfilled. Note that $h(x) > 0$ for $0 \leqslant x \leqslant 1$.

Remark 8. Concerning hypothesis (B_4) it should be noted that the property of quasi-monotonicity of the vector function (g, f) is required only in the region bounded by lower and upper solutions.

Condition 1^0 and the inequalities for $\underline{V}(x, \varepsilon)$ and $\bar{V}(x, \varepsilon)$ in condition 3^0 are satisfied for any positive κ, δ and γ. The inequalities for $\underline{U}(x, \varepsilon)$ and $\bar{U}(x, \varepsilon)$ in condition 3^0 can be fulfilled for sufficiently large κ. For example we have $\underline{U}'(0, \varepsilon) = \hat{u}'(0) + \kappa + O(\varepsilon) > 0$ for κ sufficiently large and ε sufficiently small.

Now we consider condition 2^0. Note that hypothesis (B_4) is sufficient for the proof of the inequalities (6.5) and of the corresponding inequalities for $L_\varepsilon(\underline{U}, v)$ and $L_\varepsilon(\bar{U}, v)$ in 2^0 if we establish the validity of the inequalities

$$L_\varepsilon(\underline{U}, \underline{V}) \geqslant 0 \qquad \text{for} \quad 0 < x < 1, \quad x \neq x_0, \tag{6.9}$$

$$M_\varepsilon(\underline{U}, \underline{V}) \geqslant 0 \qquad \text{for} \quad 0 < x < 1, \tag{6.10}$$

$$L_\varepsilon(\bar{U}, \bar{V}) \leqslant 0, \qquad M_\varepsilon(\bar{U}, \bar{V}) \leqslant 0 \qquad \text{for} \quad 0 < x < 1. \tag{6.11}$$

For $L_\varepsilon(\underline{U}, \underline{V})$ we get

$$\begin{aligned} L_\varepsilon(\underline{U}, \underline{V}) &= \varepsilon^2 \underline{U}'' - g(\underline{U}, \underline{V}, x, \varepsilon) \\ &= \varepsilon^2 \Big(\hat{u}''(x) - \varepsilon\beta\gamma h''(x) - \frac{1}{\varepsilon}\kappa^2 z(x, \varepsilon)\Big) - \big[\hat{g}(x) - \varepsilon\gamma h(x)(\hat{g}_u(x)\,\beta \\ &\quad + \hat{g}_v(x)) - \varepsilon \hat{g}_u(x)\, z(x, \varepsilon) + \varepsilon \hat{g}_\varepsilon(x) + O(\varepsilon^2)\big] \\ &= \varepsilon\{\gamma h(x)\, \hat{g}_u(x)[\beta - \hat{\varphi}_v(x)] + [\hat{g}_u(x) - \kappa^2]\, z(x, \varepsilon) - \hat{g}_\varepsilon(x) + O(\varepsilon)\}. \end{aligned} \tag{6.12}$$

From $\hat{g}_u(x) \geqslant 0$, (6.1) and from $h(x) > 0$ we get that the first term in the braces is greater than zero for $x \neq x_0$ and vanishes at $x = x_0$. In a small v-neighborhood of x_0 the second term is smaller than any order of ε (it is exponentially small), and the third term $-\hat{g}_\varepsilon(x)$ is positive for sufficiently small v by hypotheses (B_2). Therefore, for sufficiently small ε, the term $-\hat{g}_\varepsilon(x_0)$ determines the sign of the expression in the braces in a small

ν-neighborhood of x_0, such that we have $L_\varepsilon(\underline{U}, \underline{V}) \geqslant 0$ in this neighborhood by hypothesis (B_2). Outside this ν-neighborhood the inequality $\hat{g}_u(x) \geqslant c > 0$ holds, and this implies that the first term is dominant for sufficiently large γ. Thus, this term guarantees the validity of the inequality $L_\varepsilon(\underline{U}, \underline{V}) \geqslant 0$ outside the ν-neighborhood of x_0. Consequently, the inequality (6.9) is valid.

From $\hat{v}'' = f(\hat{u}, \hat{v}, x, 0)$ and by definition of $h(x)$ in (6.8) we get

$$\begin{aligned} M_\varepsilon(\underline{U}, \underline{V}) &= \underline{V}'' - f(\underline{U}, \underline{V}, x, \varepsilon) \\ &= \hat{v}''(x) - \varepsilon\gamma h''(x) - [\hat{f}(x) - \varepsilon\gamma(\hat{f}_u(x)\,\beta \\ &\quad + \hat{f}_v(x))\,h(x) - \varepsilon \hat{f}_u(x)\,z(x, \varepsilon) + \varepsilon \hat{f}_\varepsilon(x) + O(\varepsilon^2)] \\ &= -\varepsilon\{\gamma[h''(x) - (\hat{f}_u(x)\,\beta + \hat{f}_v(x))\,h(x)] \\ &\quad - \hat{f}_u(x)\,z(x, \varepsilon) + \hat{f}_\varepsilon(x) + O(\varepsilon)\} \\ &= \varepsilon\left\{\gamma\left[\left(\frac{\pi}{1+2\delta}\right)^2 + \hat{f}_u(x)\,\beta + \hat{f}_v(x)\right]\sin\frac{\pi(x+\delta)}{1+2\delta} + O(1)\right\}. \end{aligned} \tag{6.13}$$

Because of $\hat{f}_u(x)\,\beta + \hat{f}_v(x) > -\pi^2 + \mu$ where $\mu > 0$ (see (6.3)), and for sufficiently small δ we have

$$\left(\frac{\pi}{1+2\delta}\right)^2 + \hat{f}_u(x)\,\beta + \hat{f}_v(x) \geqslant c > 0. \tag{6.14}$$

Thus, for sufficiently large γ, the inequality (6.10) holds.

It remains to verify that the upper solution satisfies the inequalities (6.11).

By using the expressions for $\tilde{u}(x, \varepsilon)$ and $z(x, \varepsilon)$ it is not difficult to show

$$\varepsilon^2(\tilde{u}'' + \varepsilon z'') = O(\varepsilon).$$

Taking into account the relation (6.6) we obtain

$$\begin{aligned} g(\bar{U}, \bar{V}, x, \varepsilon) &= g(\hat{u}(x) + \sqrt{\varepsilon}\,\beta\gamma h(x) + O(\varepsilon), \hat{v}(x) + \sqrt{\varepsilon}\,\gamma h(x), x, \varepsilon) \\ &= g(\hat{u}(x) + \sqrt{\varepsilon}\,\beta\gamma h(x), \hat{v}(x) + \sqrt{\varepsilon}\,\gamma h(x), x, 0) + r(x, \varepsilon) \\ &= \sqrt{\varepsilon}\,\gamma h(x)(\hat{g}_u(x)\,\beta + \hat{g}_v(x)) \\ &\quad + \varepsilon\gamma^2 h^2(x)(\hat{g}_{uu}(x)\,\beta^2 + 2\hat{g}_{uv}(x)\,\beta + \hat{g}_{vv}(x)) \\ &\quad + q(x, \gamma, \varepsilon) + r(x, \varepsilon), \end{aligned}$$

here $r(x, \varepsilon)$ and $q(x, \gamma, \varepsilon)$ denote functions satisfying

$$|r(x, \varepsilon)| \leqslant c\varepsilon, \qquad |q(x, \gamma, \varepsilon)| \leqslant c_0(\gamma)\,\varepsilon^{3/2},$$

where c and $c_0(\gamma)$ are positive constants not depending of ε.

From these relations we get

$$L_\varepsilon(\bar{U}, \bar{V}) = -\sqrt{\varepsilon}\,\gamma h(x)\,\hat{g}_u(x)(\beta - \hat{\varphi}_v(x)) \\ - \varepsilon\gamma^2 h^2(x)(\hat{g}_{uu}(x)\,\beta^2 + 2\hat{g}_{uv}(x)\,\beta + \hat{g}_{vv}(x)) + q(x, \gamma, \varepsilon) + r(x, \varepsilon). \tag{6.15}$$

The first term on the right hand side of (6.15) is negative for $x \neq x_0$ and vanishes at $x = x_0$. Hence, in a small ν-neighborhood of x_0 the essential term is the second one. By condition (6.2) this term is negative in a small ν-neighborhood of x_0. Since the second and the last terms are of order $O(\varepsilon)$ and q satisfies $q(x, \gamma, \varepsilon) = o(\varepsilon)$ we may choose γ sufficiently large such that the second term determines the sign of $L_\varepsilon(\bar{U}, \bar{V})$, that is we have by (6.2) in a small ν-neighborhood of x_0

$$L_\varepsilon(\bar{U}, \bar{V}) \leqslant 0. \tag{6.16}$$

Outside the ν-neighborhood of x_0 we have $\hat{g}_u(x) \geqslant c > 0$ such that the first term of (6.15) is of order $O(\sqrt{\varepsilon})$ and dominates the other ones. For sufficiently small ε, this term guarantees the validity of (6.16) outside the ν-neighborhood of x_0 in $[0, 1]$.

Analogously to (6.13), we obtain for $M_\varepsilon(\bar{U}, \bar{V})$

$$M_\varepsilon(\bar{U}, \bar{V}) = \sqrt{\varepsilon}\,\gamma\left[-\left(\frac{\pi}{1+2\delta}\right)^2 - \hat{f}_u(x)\,\beta - \hat{f}_v(x)\right]\sin\frac{\pi(x+\delta)}{1+2\delta} + o(\sqrt{\varepsilon}).$$

By (6.14) we get from this relation that for sufficiently small ε the inequality $M_\varepsilon(\bar{U}, \bar{V}) \leqslant 0$ is fulfilled for $x \in [0, 1]$.

Therefore, the functions $(\underline{U}(x, \varepsilon), \underline{V}(x, \varepsilon))$ and $(\bar{U}(x, \varepsilon), \bar{V}(x, \varepsilon))$ constructed above satisfy the conditions 1^0–3^0 that is, they are ordered lower and upper solutions for the boundary value problem (BVP). Consequently, there exists a solution $(u(x, \varepsilon), v(x, \varepsilon))$ of (BVP) obeying the inequalities (4.3). These inequalities together with (6.7) imply that $(u(x, \varepsilon), v(x, \varepsilon))$ fulfil the relations in (6.4). This completes the proof of Theorem 2.

Remark 9. Theorem 2 is concerned with the case $v^0 < v_0(0)$, $v^1 > v_0(1)$. In other cases the stable degenerate solution is constructed analogously. For example, if we have $v^0 > v_0(0)$, $v^1 < v_0(1)$ then for the construction of $\hat{v}(x)$ with one intersection point of $\hat{v}(x)$ and $v_0(x)$ we have to use the function $\varphi_2(v, x)$ on the interval $[0, x_0]$ and the function $\varphi_1(v, x)$ on the interval $[x_0, 1]$; if we have $v^0 < v_0(0)$, $v^1 < v_0(1)$ then there may exists a stable degenerate solution such that $\hat{v}(x)$ and $v_0(x)$ have two intersection points x_1 and x_2 (say $x_1 < x_2$) and for the construction of $\hat{v}(x)$ we have to use the function $\varphi_1(v, x)$ on the intervals $[0, x_1]$ and $[x_2, 1]$ and the function $\varphi_2(v, x)$ on the interval $[x_1, x_2]$.

Remark 10. As in case of (BIVP) we get from (6.4)

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x), \qquad \lim_{\varepsilon \to 0} v(x, \varepsilon) = \hat{v}(x), \qquad 0 \leqslant x \leqslant 1.$$

Remark 11. Outside the small ν-neighborhood of x_0 we can derive the same type of asymptotic representation for the solution of (BVP) as in the case of (BIVP) in (4.9).

EXAMPLE. We consider the boundary value problem

$$\varepsilon^2 \frac{d^2u}{dx^2} = u(u - v) - \varepsilon - 9\varepsilon^2 u \equiv g(u, v, x, \varepsilon),$$

$$\frac{d^2v}{dx^2} = -9u \equiv f(u, v, x, \varepsilon), \tag{6.17}$$

$$u'(0) = u'(0) = 0, \qquad v(0) = -1, \quad v(1) = 1.$$

The smoothness assumption (C_0) is obviously satisfied. The degenerate equation $u(u - v) = 0$ has exactly two solutions, $u = \varphi_1(v, x) \equiv 0$ and $u = \varphi_2(v, x) \equiv v$, intersecting at the x-axis in the u, v, x space. Thus we have $v_0(x) \equiv 0$. Taking into account

$$g_u(\varphi_1(v, x), v, x, 0) \equiv -v \equiv g_u(\varphi_2(v, x), v, x, 0)$$

we can conclude that the hypotheses (C_1) and (C_2) hold. To check the validity of hypothesis (B_1) we consider first the boundary value problem

$$\frac{d^2v}{dx^2} = 0, \qquad v(0) = -1, \quad v(x_0) = 0, \tag{6.18}$$

where x_0 is some point in $(0, 1)$. It has the solution

$$v_1(x) \equiv \frac{x - x_0}{x_0} \tag{6.19}$$

satisfying $v_1(x) < 0$ for $0 \leqslant x < x_0$.

Next we consider the boundary value problem

$$\frac{d^2v}{dx^2} = -9v, \qquad v(x_0) = 0, \quad v(1) = 1. \tag{6.20}$$

It is easy to verify that

$$v_2(x) = \frac{\sin(3(x - x_0))}{\sin(3(1 - x_0))} \tag{6.21}$$

solves (6.20). To satisfy hypothesis (B_1) we have to determine x_0 such that

$$v_1'(x_0) = v_2'(x_0) \tag{6.22}$$

is valid. From (6.19), (6.21) and (6.22) we get the equation

$$\frac{1}{x_0} = \frac{3}{\sin(3(1 - x_0))}$$

to determine x_0. It has a unique solution x_0 in $(0, 1)$ with $x_0 \approx 0.2734$. Therefore, assumption (B_1) is valid. The degenerate stable solution $(\hat{u}(x), \hat{v}(x))$ reads

$$\hat{v}(x) \equiv \begin{cases} \dfrac{x - x_0}{x_0} & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ \dfrac{\sin(3(x - x_0))}{\sin(3(1 - x_0))} & \text{for} \quad x_0 \leqslant x \leqslant 1, \end{cases}$$

$$\hat{u}(x) \equiv \begin{cases} 0 & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ \dfrac{\sin(3(x - x_0))}{\sin(3(1 - x_0))} & \text{for} \quad x_0 \leqslant x \leqslant 1. \end{cases}$$

It is easy to verify that $(\hat{u}(x), \hat{v}(x))$ is a lower solution to (6.17), in particular, we have

$$\frac{d\hat{u}}{dx}(0) = 0, \qquad \frac{d\hat{u}}{dx}(1) = \frac{3\cos(3(1 - x_0))}{\sin(3(1 - x_0))} < 0.$$

From (6.17) we get $g_\varepsilon|_{\varepsilon=0} \equiv -1$, that means hypothesis (B_2) holds. From

$$\varphi(v, x) \equiv \begin{cases} 0 & \text{for} \quad 0 \leqslant x \leqslant x_0, \\ v & \text{for} \quad x_0 \leqslant x \leqslant 1 \end{cases}$$

it follows $\varphi_v(\hat{v}(x), x) \leqslant 1$. Using the relations

$$g_{uu} = 2, \quad g_{uv} = -1, \quad g_{vv} \equiv 0, \quad f_u = 0, \quad f_v \equiv -g$$

we have

$$\hat{g}_{uu}(x_0)\,\beta^2 + 2\hat{g}_{uv}(x_0)\,\beta + \hat{g}_{vv}(x_0) = 2\beta(\beta - 1), \qquad \hat{f}_u(x)\,\beta + \hat{f}_v(x) = -9\beta.$$

By setting $\beta = 1.01$ all conditions of hypothesis (B_3) can be satisfied. Taking into account Remark 7, it can be verified that (g, f) fulfills hypothesis (B_4). Consequently, by Theorem 2 the boundary value problem (6.17) has a solution satisfying (6.4).

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